Your Name

Your Signature

Student ID #

- Turn off all cell phones, pagers, radios, mp3 players, and other similar devices.
- This exam is closed book. You may use one $8.5'' \times 11''$ sheet of handwritten notes (both sides OK). Do not share notes. No photocopied materials are allowed.
- Graphing calculators are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. Make sure that it is easy for graders to follow what you are doing [e.g. if you perform more than one row operation in a step, label what you've done. If you perform just one row operation at a time, it should be clear what you're doing, and so the row operation does not need to be labeled in that case].
- Place $\boxed{\text{a box around your answer}}$ to each question.
- **You may write on the backs of pages** (and are expected to for some questions, in order to have enough space). Both sides of each page of the exam will be scanned.
- Raise your hand if you have a question.
- This exam has 4 pages, plus this cover sheet. Please make sure that your exam is complete.

Question	Points	Score
1	11	
2	15	
3	14	
4	10	
Total	50	

1. (11 points) **For this problem only, you do not have to show work.** For each of the following statements, circle "T" to the left if the statement is true, and "F" if the statement is false. **Here "true" means "always true".** If the are both examples of and counterexamples to the statement, the correct answer is "false." If you don't know the answer almost immediately, just make a guess and move on; time is better spent on the other exam questions.

T	F	If $\text{span}(\mathbf{v_1}, \mathbf{v_2}) = \text{span}(\mathbf{u_1}, \mathbf{u_2})$, then $\mathbf{v_1} = \mathbf{u_1}$ and $\mathbf{v_2} = \mathbf{u_2}$.
T	F	If $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ spans $\mathbb{R}^n$, then $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}$ spans $\mathbb{R}^n$.
T	F	If $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ is linearly independent, then $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}$ is linearly independent.
T	F	Suppose that one can go from the system $[A\|\mathbf{b}]$ to the system $[A'\|\mathbf{b}']$ using row operations. Then, both systems have the same solution sets.
T	F	Any linear system of equations has either 0, 1, or infinitely many solutions.
T	F	If $\mathbf{v_1} \in \text{span}(\mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4})$ then $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}$ is linearly dependent.
T	F	If $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ span $\mathbb{R}^3$, then $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ is linearly dependent.
T	F	If $[A\|\mathbf{b}]$ has infinitely many solutions, then so does $[A\|\mathbf{0}]$.
T	F	If a linear system of equations has more variables than it has equations, then it has either 0 or infinitely many solutions.
T	F	Any list containing $\mathbf{0}$ is linearly independent.
T	F	If A is a 5×3 matrix and $\text{REF}(A)$ has a pivot in every column, then for any $\mathbf{b} \in \mathbb{R}^5$, $[A\|\mathbf{b}]$ has at least one solution.

2. (15 points) Note: in this problem, you should be able to reuse your computation from part (a) to do all the other parts. Consider the vectors

$$\mathbf{u_1} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \mathbf{u_2} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}, \mathbf{u_3} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \mathbf{u_4} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}.$$

   (a) Compute the reduced echelon form of the augmented matrix $[\mathbf{u_1}\ \mathbf{u_2}\ \mathbf{u_3}\ \mathbf{u_4}|\mathbf{b}]$.

   (b) Give the solution set to the matrix equation $[\mathbf{u_1}\ \mathbf{u_2}\ \mathbf{u_3}\ \mathbf{u_4}]\mathbf{x} = \mathbf{b}$ in parametric form.

   (c) Write $\mathbf{b}$ as a linear combination of $\mathbf{u_1}$ and $\mathbf{u_2}$ (i.e. give the explicit scalar coefficients).

   (d) Is $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}$ a linearly independent list? Briefly explain how you know.

   (e) Does the list $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}$ span $\mathbb{R}^3$? Briefly explain how you know.

   (f) Is $\mathbf{u_4} \in \text{span}(\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3})$? Briefly explain how you know.

3. (14 points) Consider the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 1 & a & -1 & 0 \\ 0 & 3 & b & 0 & 0 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & d & 0 & 2 \end{array}\right].$$

For each of the following parts, **either** give an example of a specific choice of $(a,b,c,d) \in \mathbb{R}^4$ such that the given property holds **or** briefly explain why it is impossible to find such a choice of $(a,b,c,d) \in \mathbb{R}^4$.

(a) the given augmented matrix is *not* in echelon form

(b) the given augmented matrix is in *reduced* echelon form

(c) the given augmented matrix is in echelon form and has a pivot in its third column

(d) the given augmented matrix is in echelon form and has *no* pivot in its third column

(e) the given augmented matrix is in echelon form and the associated linear system has infinitely many solutions

(f) the given augmented matrix is in echelon form and the associated linear system has no solution

(g) the linear system associated to the given augmented matrix has exactly one solution

4. (10 points) Note: for this question, row operations will not be particularly helpful. Consider some vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}, \mathbf{b} \in \mathbb{R}^n$. Suppose that we know that

$$2\mathbf{v_1} + \mathbf{v_4} - \mathbf{v_3} = \mathbf{b}.$$

   (a) Give a solution to the matrix equation $[\mathbf{v_1}\ \mathbf{v_2}\ \mathbf{v_3}\ \mathbf{v_4}]\mathbf{x} = \mathbf{b}$.

   (b) Is $\mathbf{v_3} \in \operatorname{span}(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_4}, \mathbf{b})$? Explain your answer.

   (c) Suppose that we *also* know that

$$\mathbf{v_1} + \mathbf{v_2} = \frac{1}{2}\mathbf{b}.$$

   Show that the list $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}$ is linearly dependent. (Hint: think about different ways you could combine our vector equations, and which definition of linear dependence is most convenient to use here.)

   (d) (For this part, continue to assume that the equation in part (c) holds.) How many solutions does the system $[\mathbf{v_1}\ \mathbf{v_2}\ \mathbf{v_3}\ \mathbf{v_4} |\ 2\mathbf{b}]$ have? Explain how you know.